

On the influence of sliding on the Celt rattleback motion

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Abstract

The problem of the Celt stone motion with the sliding influence by the model of V.F.Zhuravlev is studied. The equations of motion are obtained and their numerical analysis is fulfilled.

1 Introduction

The classical Celt stone is a rigid convex body which moves on the horizontal plane by the gravity force near the position of the stable equilibrium. The Celt rattleback motion consists of the rotation around the vertical axis, of the vibrations around the horizontal axes and then of the rotation around the vertical axis in the opposite direction and so on [1]. This motion is possible if the body has three-axes inertia ellipsoid, one of the main central inertia axes in the equilibrium state is vertical and it cross the body surface in the contact point. Also the directions of two horizontal inertia axes do not coincide with the directions of the main surface curvature near the contact point.

Earlier in the most of works devoted to the Celt stone ([2], [3], [4]) it was supposed that the supporting point is not slide. In this assumption the equations of motion were delivered and the evolution of the angular velocity period was obtained. But the experimental data give the results which are essentially differ from the theoretical ones. The experimental period was large. As it was shown in the work about the dry friction [5] if the angular torsion velocity is non-zero then any transversal force leads to the sliding of the body. The sliding model was delivered by V.F.Zhuravlev [6] from the Coulomb law for the various types of the contact. Here we accept the simplest model with the circular contact of area and the Hertz's distribution of the contact stresses. We compare two models of the Celt stone motion namely the model without sliding and the model with sliding.

2 The sliding and the rotation friction

Let the body slides on the plane with the velocity V_1 and rotates around the vertical axis with the angular velocity Ω simultaneously. Then the horizontal friction force

\mathbf{F} and the vertical friction moment \mathbf{M} appear. In the work [6] the following friction model

$$\mathbf{F} = -F_0 \frac{\mathbf{V}}{|\mathbf{V}| + b |\varepsilon \boldsymbol{\Omega}|}, \quad \mathbf{M} = -M_0 \frac{\varepsilon \boldsymbol{\Omega}}{|\varepsilon \boldsymbol{\Omega}| + a |\mathbf{V}|} \tag{1}$$

is proposed. For the delivering of the numerical parameters in relations (1) Zhuravlev integrate the elementary Coulomb friction forces on the contact area

$$\mathbf{F} = -f \iint_S \sigma(\mathbf{r}, \vartheta) \frac{\mathbf{V} + \boldsymbol{\Omega} \times \mathbf{r}}{|\mathbf{V} + \boldsymbol{\Omega} \times \mathbf{r}|} dS, \quad \mathbf{M} = -f \iint_S \sigma(\mathbf{r}, \vartheta) \frac{\mathbf{r} \times (\mathbf{V} + \boldsymbol{\Omega} \times \mathbf{r})}{|\mathbf{r} \times (\mathbf{V} + \boldsymbol{\Omega} \times \mathbf{r})|} dS,$$

and then he use the Pade approximation.

Here $\sigma(\mathbf{r}, \vartheta)$ is the normal stresses distribution in the contact area. Further we suppose that the contact area is a circle and the stresses satisfy to the Hertz's law

$$\sigma(r) = \frac{3\mathcal{N}}{2\pi\varepsilon^3} \sqrt{\varepsilon^2 - r^2} \tag{2}$$

where \mathcal{N} is the normal compression force, and ε is the contact radius.

Under these assumptions the friction force direction is opposite to the sliding velocity direction as it is seen in relation (1).

3 The equations of motion

Now we consider both cases of motion.

Let the movable coordinate system $Cxyz$ is rigidly connected with the body, the point C is the body mass center. Let us introduce the equation of the body surface $\mathcal{F}(x, y, z) = 0$ and the unit normal \mathbf{n} at the contact point (see Fig. 1(a) and Fig. 1(b)). We introduce the turn-tensor \mathbf{P} of the body with respect to its reference position which coincides with its equilibrium state. Three independent parameters may define this tensor.

3.1 The case without sliding

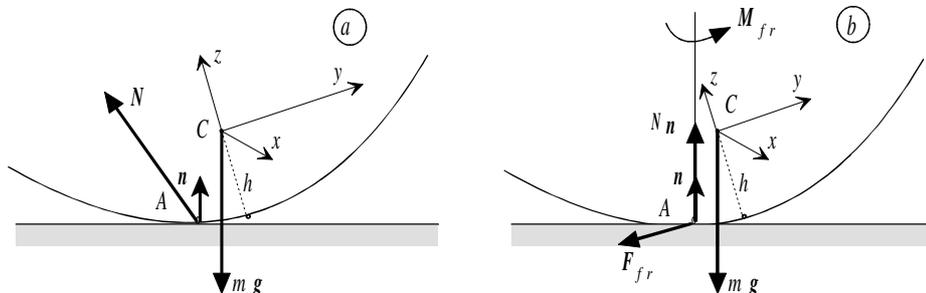


Figure 1: The coordinate system and the external forces

In the case without sliding external forces acting to the body are the gravity force $-m\mathbf{g}$ and the plane reaction \mathbf{N} which in general case is not vertical (see

Fig. 1(a)). The system of equations for this problem consists of the forces balance, of the moments balance and of the Poissons equation [7]

$$\begin{cases} m W_C = \mathbf{N} - m g \mathbf{n}, \\ \mathbf{A} \cdot \dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega} \times \mathbf{A} \cdot \boldsymbol{\Omega} = \mathbf{r}_{CA} \times \mathbf{N}, \\ \dot{\mathbf{P}} = \mathbf{P} \times \boldsymbol{\Omega}. \end{cases} \quad (3)$$

Here m is the body mass, \mathbf{A} is the body inertia tensor, $\boldsymbol{\Omega}$ is the body angular velocity. The non-sliding condition gives [1]

$$W_C = \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{AC} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{AC}) - \boldsymbol{\Omega} \times \mathbf{V}_A^*, \quad (4)$$

were \mathbf{V}_A^* is the contact point A velocity with respect to the body surface. The center mass acceleration W_C is represented by the turn-tensor $W_C(t) = W_C(\mathbf{P}(t))$. We substitute the force \mathbf{N} from the first equation (3) to the second one and obtain the differential equation for angular velocity $\boldsymbol{\Omega}$

$$\begin{aligned} (\mathbf{A} + m (\mathbf{r}_{AC}^2 \mathbf{E} - \mathbf{r}_{AC} \otimes \mathbf{r}_{AC})) \cdot \dot{\boldsymbol{\Omega}} = \\ = -\boldsymbol{\Omega} \times \mathbf{A} \cdot \boldsymbol{\Omega} + m (g \mathbf{n} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{AC}) - \boldsymbol{\Omega} \times \mathbf{V}_A^*) \times \mathbf{r}_{AC}. \end{aligned} \quad (5)$$

Here \mathbf{E} is the unit tensor of the second range, the multiplier at the $\dot{\boldsymbol{\Omega}}$ is the body inertia tensor with respect to the contact point A .

3.2 The case with sliding

The Fig 1(b) show the body rotate with sliding.

In this case the external forces acting on the body are the gravity force $-m g \mathbf{n}$, and the vertical plane reaction $\mathcal{N} \mathbf{n}$, the friction force \mathbf{F}_{fr} , and the rotating friction moment \mathbf{V}_1 (see Fig. 1(b)). The force \mathbf{F}_{fr} lies at the plane and its direction is opposite to the sliding velocity direction \mathbf{V}_1 . The moment \mathbf{M}_{fr} is vertical and opposite to the vertical component of the angular velocity $\boldsymbol{\Omega}_n = (\boldsymbol{\Omega} \cdot \mathbf{n}) \mathbf{n}$.

The differential equations similar to (3) in this case are

$$\begin{cases} m W_C = \mathcal{N} \mathbf{n} - m g \mathbf{n} + \mathbf{F}_{fr}, \\ \mathbf{A} \cdot \dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega} \times \mathbf{A} \cdot \boldsymbol{\Omega} = \mathbf{r}_{CA} \times \mathcal{N} \mathbf{n} + \mathbf{M}(\mathbf{F}_{fr}) + \mathbf{M}_{fr}, \\ \dot{\mathbf{P}} = \mathbf{P} \times \boldsymbol{\Omega}. \end{cases} \quad (6)$$

Here

$$W_C = W_1 + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{AC} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{AC}) - \boldsymbol{\Omega} \times \mathbf{V}_A^*. \quad (7)$$

The body has five degrees of freedom. To the previous unknown functions the contact point sliding is added. Here W_1 is the sliding acceleration ($W_1 \cdot \mathbf{n} = 0$).

Let us multiply the first equation (6) by the tensor-projector $\mathbf{E} - \mathbf{n} \otimes \mathbf{n}$, and obtain the two-dimensional equation for the sliding acceleration

$$m W_1 = -m \left(\dot{\boldsymbol{\Omega}} \times \mathbf{r}_{AC} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{AC}) - \boldsymbol{\Omega} \times \mathbf{V}_A^* \right) \cdot (\mathbf{E} - \mathbf{n} \otimes \mathbf{n}) + \mathbf{F}_{fr}(\mathbf{V}_1, \boldsymbol{\Omega}). \quad (8)$$

For the sliding velocity \mathbf{V}_1 in the movable coordinate system we get

$$\mathbf{W}_1 = \mathbf{P}^T \cdot (\mathbf{P} \cdot \mathbf{V}_1) \dot{=} \dot{\mathbf{V}}_1 + \boldsymbol{\Omega} \times \mathbf{V}_1.$$

After multiplying the first equation (6) by the tensor-projector $\mathbf{n} \otimes \mathbf{n}$ we find the vertical reaction

$$\mathcal{N}\mathbf{n} = m\mathbf{g}\mathbf{n} + \left(m \left(\dot{\boldsymbol{\Omega}} \times \mathbf{r}_{AC} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{AC}) - \boldsymbol{\Omega} \times \mathbf{V}_A^* \right) \cdot \mathbf{n} \right) \mathbf{n}.$$

Then the second equation (6) gives

$$\begin{aligned} & (\mathbf{A} + m\mathbf{r}_{AC2} \otimes \mathbf{r}_{AC2}) \cdot \dot{\boldsymbol{\Omega}} = \\ & = -\boldsymbol{\Omega} \times \mathbf{A} \cdot \boldsymbol{\Omega} + m \left(\mathbf{g} + (\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{AC2}) - \boldsymbol{\Omega} \times \mathbf{V}_A^*) \cdot \mathbf{n} \right) \mathbf{r}_{AC2} + \\ & \quad + \mathbf{M}(\mathbf{F}_{fr}) + \mathbf{M}_{fr}, \end{aligned} \quad (9)$$

$$\mathbf{r}_{AC2} = \mathbf{n} \times \mathbf{r}_{AC}.$$

In this case additional term to inertia tensor \mathbf{A} depends on the distance $|\mathbf{r}_{AC2}|$ between the mass center and the vertical line which crosses the body contact point.

4 The numerical analysis

We fulfill the calculations for the both models and study the dependence of the rattleback angular velocity evolution on the friction parameters.

The expressions for friction force and for the friction moment (1), in our designations are

$$\mathbf{F}_{fr} = -F_0 \frac{\mathbf{V}_1}{|\mathbf{V}_1| + b|\varepsilon\boldsymbol{\Omega}_n|}, \quad \mathbf{M}_{fr} = -M_0 \frac{\varepsilon\boldsymbol{\Omega}_n}{|\varepsilon\boldsymbol{\Omega}_n| + a|\mathbf{V}_1|}, \quad (10)$$

where $\boldsymbol{\Omega}_n = (\boldsymbol{\Omega} \cdot \mathbf{n})\mathbf{n}$ is the angular velocity of the body rotation, \mathbf{V}_1 is the sliding velocity. We suppose that the inequality

$$\boldsymbol{\Omega}_n^2 + \mathbf{V}_1^2 \neq 0 \quad (11)$$

is fulfilled during all time of motion.

We take the following friction parameters which were calculated in the paper [6]

$$F_0 = \mathcal{N}f, \quad M_0 = 3\pi\mathcal{N}f\varepsilon/16 \quad a = 15\pi/16, \quad b = 8/3\pi. \quad (12)$$

Let in the initial moment the body has the angular velocity $\boldsymbol{\Omega}_0$. We find the solution of this problem in two cases: with sliding and without sliding. We take $\varepsilon = 0.001$ and assume in the first approximation that $\mathcal{N} = m\mathbf{g}$.

It is well-known that the classical Celt stone changes the direction of its rotation during the motion. Fig. 2 shows the dependence of the rotating angular velocity on time for the case without sliding (at the left side) and for the case with sliding for various friction coefficients $f = 0.35$, $f = 0.25$, and $f = 0.15$. It is seen that if the friction coefficient f decreases then the period of the Celt effect increases, also decreases the energy of motion (the angular velocity). For the accepted parameters

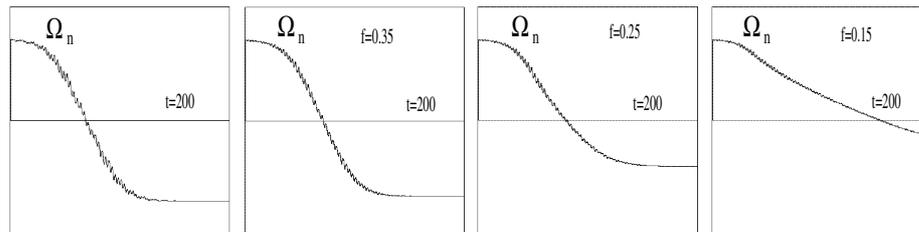


Figure 2: The influence of sliding to the rotating velocity

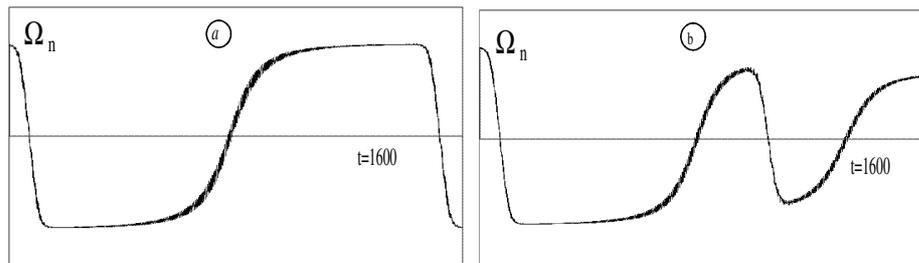


Figure 3: The angular velocity evolution

for $f < 0.1$ the Celt effect does not exist. The Fig. 3 illustrates the Celt motion during the comparatively long time. Here the rotating angular velocity changes its sign more than two times. The left side of the graphics 3(a) shows the classical case without sliding and the right side 3(b) shows the motion with sliding with the friction coefficient $f = 0.35$. After the second changing of the velocity sign these graphics are essentially differ and further the angular velocity in the case with sliding changes its sign more quickly.

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